Single Phase Series AC Circuits

![Diagram of a single phase series AC circuit with impedances R1, L1, R2, L2, R3, and a capacitor C.]
Single Phase Series AC Circuits
(Much of this material has come from Electrical & Electronic Principles & Technology by John Bird)

Purely Resistive Circuit

Diagram shows a purely resistive circuit together with current and voltage waveforms and the phasor diagrams. Since the current and voltage are in phase it follows that we can continue to use the familiar Ohms Law, ie \( V = IR \).

Purely Inductive Circuit

With a purely inductive circuit we see \( I_L \) lags \( V_L \) by 90°.

In a purely inductive circuit the opposition to a.c. current flow is called Inductive Reactance and denoted \( X_L \), it has units of ohms.

Not proved here, but the relationship between inductive reactance and current is

\[
X_L = \frac{V_L}{I_L} = 2\pi fL
\]

Where \( f \) is the frequency in Hz and \( L \) the inductance in henries.

Since \( f = \frac{\omega}{2\pi} \) we can also have

\[
X_L = \omega L
\]

It should be noted that \( X_L \) is proportional to frequency as shown below.
Worked Example
A coil has an inductance of 40mH and negligible resistance. Calculate its inductive reactance and resulting current if connected to a) a 240v, 50Hz supply and b) a 100v, 1kHz supply.

a) The inductive reactance \( X_L = 2\pi fL = 2\pi \times 50 \times 40 \times 10^{-3} = 12.56\Omega \)

Since \( X_L = \frac{V_L}{I_L} \) it follows that \( I_L = \frac{V_L}{X_L} = \frac{240}{12.56} = 19.1\text{A} \)

If you want, you can solve b)

Purely Capacitive Circuit

In a purely capacitive circuit we see that the current leads the voltage by 90°.

In a purely capacitive circuit the opposition to the flow of a.c. current is called Capacitive Reactance and denoted \( X_C \) and has units of ohms.

Again, not proved here but capacitive reactance is given by

\[ X_C = \frac{V_C}{I_C} = \frac{1}{2\pi fC} = \frac{1}{\omega C} \]

\( X_C \) also varies with frequency as shown below.
Worked Example

Determine the capacitive reactance of a capacitor of 10µF when connected to a circuit of frequency (a) 50Hz and (b) 20Hz.

(a) Capacitive reactance

\[
X_C = \frac{1}{\omega C} = \frac{1}{2\pi f C} = \frac{1}{2\pi \times 50 \times 10^{-6}} = 318.3\Omega
\]

You can solve (b) if you want to.

Note:
The relationship between voltage and current for inductive and capacitive circuits can be remembered using CIVIL.

Which is – In a capacitor (C) the current (I) is ahead of the voltage (V) and the voltage (V) is ahead of current (I) for an inductor (L).

R-L Series AC Circuits

The first thing to note is that the applied voltage (V) is not the arithmetic sum of the voltages across the resistor and inductor but the phasor sum of \(V_R\) and \(V_L\).

Secondly, we see the that the current I lags the applied voltage V by an angle \(\phi\) somewhere between 0 and 90°.

It follows that \(V = \sqrt{V_R^2 + V_L^2}\) and that \(\tan \phi = \frac{V_L}{V_R}\)

Next, in an a.c. circuit the ratio \(\frac{V}{I}\) is called Impedance and denoted \(Z\) so \(Z = \frac{V}{I}\) so we can write \(V = IZ\)

We also have \(X_L = \frac{V_L}{I}\) so can have \(V_L = IX_L\)
Similarly, \( V_R = IR \)

Notice that these have been added to the voltage triangle. If we now divide each term on the velocity triangle by I, we obtain an Impedance Triangle.

From the impedance triangle, we can obtain the following relationships:

\[
Z = \sqrt{R^2 + X_L^2}
\]

\[
\tan \phi = \frac{X_L}{R}
\]

\[
\sin \phi = \frac{X_L}{Z}
\]

\[
\cos \phi = \frac{R}{Z}
\]

**Worked Example 1**

A pure inductance of 1.273mH is connected in series with a pure resistance of 30Ω. If the frequency of the sinusoidal supply is 5kHz and the p.d. across the 30Ω resistor is 6v, determine the supply voltage and the voltage across the 1.273mH inductor. Draw the phase diagram.

We have:

- \( L = 1.273 \text{mH} \)
- \( R = 30 \Omega \)
- \( f = 5 \text{kHz} \)
- \( V_R = 6 \text{v} \)

We want:

- \( V \)
- \( V_L \)

Supply voltage \( V = IZ \)

Current \( I = \frac{V_R}{R} = \frac{6}{30} = 0.20 \text{A} \)

Impedance \( Z = \sqrt{R^2 + X_L^2} \)

Where \( X_L = 2\pi fL = 2\pi \times 5 \times 10^3 \times 1.273 \times 10^{-3} = 40 \Omega \)

So \( Z = \sqrt{30^2 + 40^2} = 50 \Omega \)

Therefore, supply voltage \( V = 0.20 \times 50 = 10 \text{v} \)

Now inductive reactance \( X_L = \frac{V_L}{I_L} \) so rearranging we have \( V_L = X_L I_L = 40 \times 0.2 = 8 \text{v} \)

For the phase diagram see below.
Worked Example 2
A coil of inductance 159.2mH and resistance 20Ω is connected in series with a 60Ω resistor to a 240v, 50Hz supply. Determine:

(a) The impedance of the circuit
(b) The current in the circuit
(c) The circuit phase angle
(d) The p.d. across the 60Ω resistor
(e) The p.d across the coil

Draw the phasor diagram

It should be noted that when impedances are connected in series the individual resistances may be added to give a total resistance, hence included with the original circuit diagram is the equivalent circuit diagram.

\[
Z = \sqrt{R^2 + X_L^2}
\]

\[R = 80Ω\]

\[X_L = 2\pi fL = 2\pi \times 50 \times 159.2 \times 10^{-3} = 50Ω\]

Therefore \(Z = \sqrt{80^2 + 50^2} = 94.34Ω\)

b) Circuit current \(I\)

\[V = IZ \text{ therefore } I = \frac{V}{Z} = \frac{240}{94.3} = 2.54A\]

c) Phase angle \(\phi\)
\[
\tan \phi = \frac{X_L}{R} \text{ therefore } \phi = \tan^{-1} \left( \frac{X_L}{R} \right) = \tan^{-1} \frac{50}{80} = 32^\circ \text{ lagging}
\]

d) P.d across 60Ω resistor

\[V_R = IR = 2.54 \times 60 = 152.4v\]

e) P.d across the coil

\[V_{coil} = IZ_{coil}\]

Where \[Z_{coil} = \sqrt{R_{coil}^2 + X_L^2} = \sqrt{20^2 + 50^2} = 53.85\Omega\]

Therefore \[V_{coil} = 2.54 \times 53.85 = 136.78v \text{ say 137v}\]

For the phasor diagram the supply voltage will be the vector sum of the voltage across the resistor (\(V_R\)) and the voltage across the coil (\(V_{coil}\)), where the voltage across the coil (\(V_{coil}\)) will be the vector sum of the voltage across the inductor (\(V_L\)) and the voltage across its ‘internal’ resistance (\(V_{Rcoil}\)). We do not have \(V_L\) or \(V_{Rcoil}\), therefore these need to be calculated before the phasor can be drawn.

\[V_L = IX_L = 2.54 \times 50 = 127v\]

\[V_{Rcoil} = IR_{coil} = 2.54 \times 20 = 50.8v\]
R-C Series AC Circuit

In the above phasor diagram, we see $V_R$ in phase with the current $I$ while $V_C$ lags $V_R$ by $90^\circ$.

$V$ is the phasor sum of $V_R$ and $V_C$, and the current $I$ leads the applied voltage $V$ by an angle $\alpha$, lying between $0^\circ$ and $90^\circ$.

Also shown are the velocity triangle together with the impedance triangle.

The relevant equations developed from the two triangles are:

\[
V = \sqrt{V_R^2 + V_C^2}
\]

\[
\tan \alpha = \frac{V_C}{V_R}
\]

\[
Z = \sqrt{R^2 + X_C^2}
\]

\[
\tan \alpha = \frac{X_C}{R}
\]

\[
\sin \alpha = \frac{X_C}{Z}
\]

\[
\cos \alpha = \frac{R}{Z}
\]

**Worked Example**

A capacitor $C$ is connected in series with a $40\Omega$ resistor across a supply of frequency $60$ Hz. A current of $3A$ flows in the circuit of impedance $50\Omega$. Calculate:

(a) The value of capacitance
(b) The supply voltage
(c) The phase angle between supply voltage and current
(d) The p.d. across the resistor
(e) The p.d across the capacitor
Draw the phasor diagram.

We have
R = 40Ω
f = 60Hz
I = 3A
Z = 50Ω

(a) The capacitance?

\[ Z = \sqrt{R^2 + X_C^2} \text{ so, re arranging we have } X_C = \sqrt{Z^2 - R^2} = \sqrt{50^2 - 40^2} = 30Ω \]

\[ X_C = \frac{1}{2\pi fC} \text{ which rearranged gives } C = \frac{1}{\frac{1}{2\pi fC}} = \frac{1}{\frac{1}{2\pi \times 60 \times 30}} = 8.842 \times 10^{-5} = 88.42µF \]

(b) \( V = IZ = 3 \times 50 = 150v \)

(c) \( \tan \alpha = \frac{X_C}{R} \text{ therefore, } \alpha = \tan^{-1}\frac{X_C}{R} = \tan^{-1}\frac{30}{40} = 36.86° \)

(d) \( V_R = IR = 3 \times 40 = 120v \)

(e) \( V_C = IX_C = 3 \times 30 = 90v \)

The phasor diagram is:

\[ V_R = 120v \quad I = 3A \]

\[ V_C = 90v \]

\[ V = 150v \]

\[ 36.86° \]
R-L-C Series AC Circuit

The applied voltage $V$ is the phasor sum of $V_R$, $V_L$ and $V_C$.

$V_L$ and $V_C$ are anti-phase (displaced by $180^\circ$) and there are 3 possible phasor diagrams depending in the relative values of $V_L$ and $V_C$.

It will be noticed in (d) above that when $X_L = X_C$, the applied voltage $V$ and the current $I$ are in phase. When this is the case it is called series resonance.

The additional equations here are:

When $X_L > X_C$

$$Z = \sqrt{R^2 + (X_L - X_C)^2}$$

$$\tan \phi = \frac{X_L - X_C}{R}$$

When $X_C > X_L$

$$Z = \sqrt{R^2 + (X_C - X_L)^2}$$

$$\tan \alpha = \frac{X_C - X_L}{R}$$
Worked Example
A coil of resistance 5Ω and inductance 120mH in series with a 100µF capacitor is connected to a 300v, 50Hz supply. Calculate:
(a) The current flowing
(b) The phase difference between supply voltage and current
(c) The voltage across the coil
(d) The voltage across the capacitor
Draw the phasor diagram

The circuit is:

We have:
R = 5Ω
L = 120mH
C = 100µF
V = 300v
f = 50Hz

(a) Current flowing?
V = IZ

\[ X_L = 2\pi fL = 2\pi \times 50 \times 120 \times 10^{-3} = 37.7Ω \]
\[ X_C = \frac{1}{2\pi fC} = \frac{1}{2\pi \times 50 \times 100 \times 10^{-6}} = 31.83Ω \]

Therefore \( X_L > X_C \) and \( Z = \sqrt{R^2 + (X_L - X_C)^2} = \sqrt{5^2 + (37.7 - 31.83)^2} = 7.71Ω \)

Therefore \( I = \frac{V}{Z} = \frac{300}{7.71} = 38.91A \)

(b) Phase difference?
\[ \tan \phi = \frac{X_L - X_C}{R} \text{ therefore } \phi = \tan^{-1} \left( \frac{X_L - X_C}{R} \right) = \tan^{-1} \left( \frac{37.7 - 31.83}{5} \right) = 49.58° \]

(c) \( V_{coil} =? \)
\[ V_{coil} = IZ_{coil} \]

Where \( Z_{coil} = \sqrt{R^2 + X_L^2} = \sqrt{5^2 + 37.7^2} = 38.03Ω \)
\[ V_{\text{coil}} = 38.91 \times 38.03 = 1479.75 \text{v} \]

Phase angle of coil \( \phi = \tan^{-1} \frac{X_L}{R} = \tan^{-1} \frac{37.7}{5} = 82.45^\circ \text{ lagging} \)

(d) \( V_C = ? \)

\[ V_C = I X_C = 38.91 \times 31.83 = 1238.5 \text{v} \]

**Series Connected Impedances**

For series-connected impedances the total circuit impedance can be represented as a single L-C-R circuit by combining all values of resistance together, all values of inductance together and all values of capacitance together. Not forgetting that \( \frac{1}{C} = \frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3} + \cdots \). This is illustrated in the diagram below.
Tutorial Problems

1. An alternating voltage given by \( v = 100\sin 240t \) volts is applied across a coil of resistance 32\( \Omega \) and inductance 100 mH. Determine (a) the circuit impedance, (b) the current flowing, (c) the p.d. across the resistance and (d) the p.d. across the inductance.
   \((40\Omega, 1.77A, 56.64v, 42.48v)\)

2. A coil of inductance 636.6mH and negligible resistance is connected in series with a 100\( \Omega \) resistor to a 250v, 50Hz supply. Calculate (a) the inductive reactance of the coil, (b) the impedance of the circuit, (c) the current in the circuit, (d) the p.d. across each component, and (e) the circuit phase angle.
   \((200\Omega, 223.6\Omega, 1.118A, 111.8v, 223.6v, 63.43^o lagging)\)

3. A 24.87\( \mu \)F capacitor and a 30\( \Omega \) resistor are connected in series across a 150v supply. If the current flowing is 3A, determine (a) the frequency of the supply, (b) the p.d. across the resistor and (c) the p.d. across the capacitor.
   \((160Hz, 90v, 120v)\)

4. An alternating voltage \( v = 250\sin 800t \) volts is applied across a series circuit containing a 30\( \Omega \) resistor and 50\( \mu \)F capacitor. Calculate (a) the circuit impedance, (b) the current flowing, (c) the p.d. across the resistor, (d) the p.d. across the capacitor and (e) the phase angle between the voltage and current.
   \((39.05\Omega, 4.526A, 135.78v, 113.15v, 39.8^o leading)\)

5. A 40\( \mu \)F capacitor in series with a coil of resistance 8\( \Omega \) and inductance 80mH is connected to a 200v, 100Hz supply. Calculate (a) the circuit impedance, (b) the current flowing, (c) the phase angle between voltage and current, (d) the voltage across the coil and (e) the voltage across the capacitor.
   \((13.18\Omega, 15.17A, 52.64 lagging, 772.19v, 603.6v)\)

6. Three impedances are connected in series across a 100v, 2kHz supply. The impedances comprise:

   i. An inductance of 0.45mH and 2\( \Omega \) resistance
   ii. An inductance of 570\( \mu \)H and 5\( \Omega \) resistance
   iii. A capacitor of capacitance 10\( \mu \)F and resistance 3\( \Omega \)

Assuming no mutual inductive effects between the two inductances, calculate (a) the circuit impedance, (b) the circuit current, (c) the circuit phase angle and (d) the voltage across each impedance.
   \((11.12\Omega, 8.99A, 25.92^o lagging, 53.92v,78.53v, 76.46v)\)
**Series Resonance**

In the section dealing with R-L-C circuits it was mentioned that when $X_C = X_L$ we had a situation where the applied voltage ($V$) and the current ($I$) were in phase and the effect is called series resonance.

When resonance occurs:

$V_L = V_C$

$Z = R$, and is the minimum impedance possible in an R-L-C circuit.

$I = V/R$ and is the maximum current possible in an R-L-C circuit

Since $X_C = X_L$ it is possible to calculate a resonance frequency $f_r$

$$2\pi f_L = \frac{1}{2\pi f_C}$$

Therefore $f^2 = \frac{1}{(2\pi)^2 L^2 C^2}$

Or $f_r = \frac{1}{2\pi \sqrt{LC}}$

**Q-factor**

At resonance, if $R$ is small compared with $X_L$ and $X_C$, it is possible for $V_L$ and $V_C$ to be many times greater than the applied voltage and this results in a term call Voltage Magnification.

Voltage Magnification at Resonance = $\frac{\text{voltage across } L \text{ (or } C)}{\text{supply voltage}}$

The ratio is a measure of the quality of the circuit (as a resonator or tuning device) and is also known as the Q-factor.

Not proved here but $Q$-factor $= \frac{2\pi f_r L}{R} = \frac{1}{\sqrt{LC}} \left(\frac{L}{R}\right) = \frac{1}{R} \sqrt{\frac{L}{C}}$

**Worked Example**

A coil of negligible resistance and inductance 100mH is connected in series with a capacitance of 2µF and a resistance of 10Ω across a 50v, variable frequency supply. Determine (a) the resonant frequency, (b) the current at resonance, (c) the voltage across the coil and the capacitor at resonance and (d) the Q-factor of the circuit.

1) $f_r = \frac{1}{2\pi \sqrt{LC}} = \frac{1}{2\pi \sqrt{100 \times 10^{-3} \times 2 \times 10^{-6}}} = 355.9$Hz

2) $I = \frac{V}{R} = \frac{50}{10} = 5$A

3) $V_L = I X_L = I \times 2\pi f_r L = 5 \times 2\pi \times 355.9 \times 100 \times 10^{-3} = 1118$V

This also equals $V_C$, which could have been calculated using $V_C = I X_C$
4) \[ Q \text{-factor} = \frac{V_L}{V} \text{ or } \frac{V_C}{V} = \frac{1118}{50} = 22.36 \]

**Bandwidth**

The above diagram shows how I varies with frequency in an R-L-C series circuit. We see that at resonance the current has a maximum denoted \( I_r \). On the diagram the point A and B occur where the current is 0.707 of the maximum. The frequencies that correspond to this are \( f_1 \) and \( f_2 \).

Now the power delivered to the circuit will be \( I^2R \)

At \( I = 0.707I_r \) the power is \((0.707I_r)^2 R = 0.5I_r^2 R\), in other words its half the power at \( f_r \).

The point corresponding to \( f_1 \) and \( f_2 \) are called the half-power points and the distance between them, \((f_2 - f_1)\) is called the bandwidth.

Not proved here but it can be shown that:

\[ Q = \frac{f_r}{(f_2 - f_1)} \text{ or } (f_2 - f_1) = \frac{f_r}{Q} \]

**Power in a.c. Circuits**

Purely Resistive Circuit – the average power dissipated is given by \( P = IV = I^2R = V^2/R \) watts, where \( V \) and \( I \) are the r.m.s values.

For purely inductive and purely capacitive a.c. circuits, the average power is zero.

For an R-L, R-C or R-L-C series a.c. circuit the average power is given by \( P = VI\cos\phi \text{ or } P = I^2R \text{ watts, where } V \text{ and } I \) are r.m.s. values.

It is also possible to define an apparent power \( S = VI \) which has units of voltamperes.

A power factor is also defined as \( \text{Power factor} = \frac{\text{true power } P}{\text{apparent power } S} = \frac{VI\cos\phi}{VI} = \cos\phi = \frac{Z}{R} \)
**Worked Example 1**

A series circuit of resistance 60Ω and inductance 75mH is connected to a 110v, 60Hz supply. Calculate the power dissipated.

\[ P = I^2R \]

So, we need to determine I and we know \( V =IZ \)

Where \( Z = \sqrt{R^2 + X_L^2} \)

And \( X_L = 2\pi fL = 2\pi \times 60 \times 75 \times 10^{-3} = 28.27Ω \)

Therefore \( Z = \sqrt{60^2 + 28.27^2} = 66.33Ω \)

Thus \( I = \frac{V}{Z} = \frac{110}{66.33} = 1.658A \)

And therefore \( P = 1.658^2 \times 60 = 165W \)

Alternatively use \( P = VI\cos\phi \) where \( \cos\phi = \frac{R}{Z} \)

**Tutorial Problems**

1) A coil of 0.5H inductance and 8Ω resistance is connected in series with a capacitor across a 200v, 50Hz supply. If the current is in phase with the supply voltage, determine the capacitance and the p.d. across its terminals.

\( (20.26\mu F, 3.927kv) \)

2) An 80Ω resistor and 6µF capacitor are connected in series across a 150v, 200Hz supply. Calculate (a) the circuit impedance, (b) the current flowing and (c) the power dissipated in the circuit.

\( (154.9Ω, 0.971A, 75.43W) \)

3) A coil of resistance 25Ω and inductance 100mH is connected in series with a capacitor of 0.12µF across a 200v, variable frequency supply. Calculate (a) the resonant frequency, (b) the current at resonance and (c) the factor by which the voltage across the reactance is greater than the supply voltage.

\( (1.453kHz, 8A, 36.51) \)
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